Mixed Fractional Differentiation Operators in Mixed Weighted Hölder Spaces

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Abstract:
We study mixed fractional derivative of functions of two variables in weighted Hölder spaces of different orders in each variable. The obtained results extend the well-known theorem of Hardy-Littlewood for the one-dimensional fractional derivative to the case of mixed Hölderness.
Keywords: *functions of two-variables, derivative in Marchaud form, mixed differences, mixed fractional integral, weight, weighted Hölder spaces.*

1. Introduction

The mapping properties of the one-dimensional fractional Riemann-Liouville operator $\left(I_{a+}^{\alpha} f\right)(x)=\Gamma^{-1}(\alpha)\left(t_{+}^{\alpha-1} * f\right)(x),\ x>a$, are well studied both in weighted Hölder spaces oe in generalized Hölder spaces. A non-weighted statement on the action of the fractional integral operator from H_0^{λ} into $H_0^{\lambda+\alpha}$ is due to Hardy and Littlewood ([1], see [6], Theorems 3.1 and 3.2), and it is known that the operator I_{a+}^{α} with $0<\alpha<1$ establishes an isomorphism between the Hölder spaces $H_0^{\lambda}([a,b])$ and $H_0^{\lambda+\alpha}([a,b])$ of functions vanishing at the point $x=a$, if $\alpha+\lambda<1$. The weighted results with power weights were obtained in [6] (see Theorems 3.3, 3.4 and 13.13). For weighted generalized Hölder spaces $H_0^{\omega}(\rho)$ of functions φ with a given dominant of continuity modulus of $\rho\varphi$, mapping properties in the case of power weight were studied see [4], [5], [7], see also their presentation in [6], Section 13.6. Different proofs were suggested in [2], [3], where the case of complex fractional orders was also considered, the shortest proof is given in [2]. A detailed review of these and some other similar results can be found in [6].

In the multidimensional case, the statement about the properties of a map in Hölder spaces for a mixed fractional Riemann – Liouville integral was studied in [8], [9], [12], [15], [16], [17], [18]. When weighted generalized Hölder spaces see [13], [14].

Mixed fractional derivatives were not studied in the weighted Hölder spaces. Mixed fractional derivatives were studied when non-weighed Hölder spaces see [9], [10], [11], [12], [18]. This paper is devoted to the study of the properties of a map in weighted Hölder spaces.

We consider the operator mixed fractional derivatives in the rectangle $Q=\{(x,y): 0<x<b,\ 0<y<d\}$.

2. Preliminaries

2.1.Notation and a technical lemma.

For a continuous function $\varphi(x,y)$ on $\mathbf{R}^{2,}$ we introduce the notation

$$\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)=\varphi(x+h,y)-\varphi(x,y),$$

$$\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)=\varphi(x,y+\eta)-\varphi(x,y),$$

$$\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)=\varphi(x+h,y+\eta)-\varphi(x,y+\eta)-\varphi(x+h,y)+\varphi(x,y)$$

so that

$$\varphi(x+h,y+\eta)=\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)+\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)+ +\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)+\varphi(x,y). \quad (2)$$

Everywhere in the sequel by C, C_1, C_2 etc., we denote positive constants which may different values in different occurrences and even in the same line.

We introduce two types of mixed Hölder spaces by the following definitions.

Definition 1. Let $\lambda,\gamma\in(0,1]$. We say that $\varphi\in H^{\lambda,\gamma}(Q)$ if

$$|\varphi(x_1,y_1)-\varphi(x_2,y_2)|\le C_1|x_1-x_2|^{\lambda}+C_2|y_1-y_2|^{\gamma} \quad (3)$$

for all $(x_1,y_1),(x_2,y_2)\in Q$. Condition (3) is equivalent to the coupe of the separate conditions

$$\left|\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)\right|\le C_1|h|^{\lambda}, \quad \left|\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)\right|\le C_2|\eta|^{\gamma} \quad (4)$$

uniform with respect to another variable. By $H_0^{\lambda,\gamma}(Q)$ we define a subspace of functions $f\in H^{\lambda,\gamma}(Q)$, vanishing at the boundaries $x=0$ and $y=0$ of Q.

Let $\lambda=0$ and\or $\gamma=0$. We put $H^{0,0}(Q)=L^{\infty}(Q)$ and

$$H^{\lambda,0}(Q)=\left\{\varphi\in L^{\infty}(Q):\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right|\le C_1|h|^{\lambda}\right\},\ \lambda\in(0,1]$$

$$H^{0,\gamma}(Q)=\left\{\varphi\in L^{\infty}(Q):\left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right|\le C_2|\eta|^{\gamma}\right\},\ \gamma\in(0,1].$$

Definition 2. We say that $\varphi(x,y)\in\widetilde{H}^{\lambda,\gamma}(Q)$, where $\lambda,\gamma\in(0,1]$, if

$\varphi\in H^{\lambda,\gamma}(Q)$ and $\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C_3|h|^{\lambda}|\eta|^{\gamma}$.

We say that $\varphi\in\widetilde{H}_0^{\lambda,\gamma}(Q)$, if $\varphi(x,y)\in\widetilde{H}^{\lambda,\gamma}(Q)$ and $\varphi(0,y)\equiv\varphi(x,0)\equiv0$.

These spaces become Banach spaces under the standard definition of the norms:

$$\|\varphi\|_{H^{\lambda,\gamma}}:=\|\varphi\|_{C(Q)}+\sup_{\substack{x,x+h\in[0,b]\\ y\in[0,d]}}\frac{\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right|}{|h|^{\lambda}}+$$

$$+\sup_{\substack{y,y+\eta\in[0,d]\\ x\in[0,b]}}\frac{\left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right|}{|\eta|^{\gamma}},$$

$$\|\varphi\|_{\widetilde{H}^{\lambda,\gamma}}:=\|\varphi\|_{H^{\lambda,\gamma}}+\sup_{\substack{x,x+h\in[0,b]\\ y,y+\eta\in[0,d]}}\frac{\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|}{|h|^{\lambda}|\eta|^{\gamma}}.$$

Note that

$$\varphi\in H^{\lambda,\gamma}(Q)\Rightarrow\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C_\theta|h|^{\theta\lambda}|\eta|^{(1-\theta)\gamma}\qquad(5)$$

for any $\theta\in[0,1]$, where $C_\theta=2C_1^{\theta}C_2^{1-\theta}$, so that

$$\bigcap_{0\le\theta\le1}\widetilde{H}^{\theta\lambda,\gamma(1-\theta)}(Q)\hookrightarrow H^{\lambda,\gamma}(Q)\hookrightarrow\widetilde{H}^{\lambda,\gamma}(Q)\qquad(6)$$

where $\hookrightarrow$ stands for the continuous embedding and the norm for $\bigcap_{0\le\theta\le1}\widetilde{H}^{\theta\lambda,\gamma(1-\theta)}(Q)$ is introduced as the maximum in θ of norms for $\widetilde{H}^{\theta\lambda,\gamma(1-\theta)}(Q)$. Since $\theta\in[0,1]$ is arbitrary, it is not hard to see that the inequality in (5) is equivalent (up to the constant factor C) to

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C\min\left\{|h|^{\lambda}|\eta|^{\gamma}\right\}.\qquad(7)$$

We will also make use of the following weighted spaces. Let $\rho(x,y)$ be a non-negative function on Q (we will only deal with degenerate weights $\rho(x,y)=\rho(x)\rho(y)$).

Definition 3. By $H^{\lambda,\gamma}(Q;\rho)$ and $\widetilde{H}^{\lambda,\gamma}(Q;\rho)$ we denote the spaces of functions $\varphi(x,y)$ such that $\rho\varphi\in H^{\lambda,\gamma}(Q)$, $\rho\varphi\in\widetilde{H}^{\lambda,\gamma}(Q)$, respectively, equipped with the norms

$\|\varphi\|_{H^{\lambda,\gamma}(Q;\rho)}=\|\rho\varphi\|_{H^{\lambda,\gamma}(Q)}$, $\|\varphi\|_{\widetilde{H}^{\lambda,\gamma}(Q;\rho)}=\|\rho\varphi\|_{\widetilde{H}^{\lambda,\gamma}(Q)}$.

By $H_0^{\lambda,\gamma}(Q;\rho)=H_0^{\lambda,\gamma}(\rho)$ and $\widetilde{H}_0^{\lambda,\gamma}(Q;\rho)=\widetilde{H}_0^{\lambda,\gamma}(\rho)$ we denote the corresponding subspaces of functions φ such that $\rho\varphi|_{x=0}\equiv\rho\varphi|_{y=0}\equiv0$.

Below we follow some technical estimations suggested in [2] for the case of one-dimensional Riemann-Liouville fractional integrals. We denote

$$B(x,y;t,s)=\frac{\rho(x,y)-\rho(t,s)}{\rho(t,s)(x-t)^{1+\alpha}(y-s)^{1+\beta}},\qquad(8)$$

where $0<\alpha,\beta<1, 0<t<x<b,\ 0<s<y<d$ and

$$B_1(x,t)=\frac{\rho(x)-\rho(t)}{\rho(t)(x-t)^{1+\alpha}}\quad B_2(y,s)=\frac{\rho(y)-\rho(s)}{\rho(s)(y-s)^{1+\beta}}.\qquad(9)$$

In the case $\rho(x,y)=\rho(x)\rho(y)$ we have

$$B(x,y;t,s)=B_1(x,t)B_2(y,s)+\frac{B_1(x,t)}{(y-s)^{1+\beta}}+\frac{B_2(y,s)}{(x-t)^{1+\alpha}}.$$

Let also

$D_1(x,h,t)=B_1(x+h,t)-B_1(x,t)$, $t,x,x+h\in[0,b], h>0$,

$D_2(y,\eta,s)=B_2(y+\eta,s)-B_2(y,s)$, $s,y,y+\eta\in[0,d],\eta>0$.

Lemma 1. (see [2]). Let $\rho(x)=x^{\mu}$, $\mu\in R^1$, $0<\alpha<1$. Then

$$|B_1(x,t)|\le C\left(\frac{x}{t}\right)^{\max(\mu-1,0)}\frac{1}{t(x-t)^{\alpha}}\qquad(10)$$

$$|D_1(x,h,t)|\le C\left(\frac{x}{t}\right)^{\max(\mu-1,0)}\frac{h}{t(x+h-t)(x-t)^{\alpha}}.\qquad(11)$$

Similar estimates hold for $B_2(y,s)$ and $D_2(y,\eta,s)$ with $\rho(y)=y^{\nu}$.

Remark 1. All the weighted estimations of functional integrals in the sequel are based on inequalities (10)-(11). Note that the right-hand sides of these inequalities have the exponent $\max(\mu-1,0)$, which means that in the proof, it suffices to consider only the case $\mu\ge1$, evaluations for $\mu<1$ being the same as for $\mu=1$.

2.2. A one-dimensional statement

The following statement is known (see the presentation of this proof in [6], p. 190); a shorter proof was given in [2]. Nevertheless, we recall the scheme of the proof from [2] to make the presentation easier for the two-dimensional case.

Let $\rho(x)$ the weight function and put it $\psi(x)=\rho(x)\varphi(x)$ $\varphi(x)\in H_0^{\lambda}([0,b];\rho)$. Evidently $\psi(x)\in H^{\lambda}([0,b])$ and $\psi(0)=0$. It is easy to see that

$$\left(\rho I_{0+}^{\alpha}\varphi\right)(x)=\left(I_{0+}^{\alpha}\psi\right)(x)+\left(J_{0+}^{\alpha}\psi\right)(x),\qquad(13)$$

where $\left(J_{0+}^{\alpha}\psi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x B_1(x,t)\psi(t)dt$ so that in (13) we have the farctional integral if $0<\alpha<1$ and the fractional derivative if $-1<\alpha<0$.

The representation (13) for the fractional (integral) derivative shows that the estimate for the continuity modulus in the weighted case reduces to two simpler estimates:

1) the known non-weighted estimate of Hardy-Littlewood for fractional integral and fractional derivative;

2) the estimate of the second term in (13), which is the main part of the job.

Theorem 1. Let $0<\alpha<1$, $\rho(x)=x^{\mu}$ and $|\psi(x)|\le Cx^{\alpha+\lambda}$ with $\mu<1+\lambda$. Then the operator $\left(J_{0+}^{-\alpha}\psi\right)(x)\in H^{\lambda}([0,1])$, $\lambda+\alpha<1$ and $\left\|\left(J_{0+}^{-\alpha}\psi\right)(x)\right\|_{\lambda}\le C\sup_{x\in[0,1]}\left|x^{-\lambda-\alpha}\psi(x)\right|$.

Proof. In the proof, we use the following notation

$$\left(J_{0+}^{-\alpha}\psi\right)(x+h)-\left(J_{0+}^{-\alpha}\psi\right)(x)=F_1(x,h)+F_2(x,h), \qquad (14)$$

where

$$F_1(x,h)=\int_x^{x+h}B(x+h,t)\psi(t)dt\,,$$
$$F_2(x,h)=\int_0^{x}D(x,h,t)\psi(t)dt\,. \qquad (15)$$

The estimate of F_1 the case $1\le\mu<\lambda+1$. The estimate (10) $B(x,t)$ implies

$$|F_1|\le C(x+h)^{\mu-1}\int_x^{x+h}t^{\lambda+\alpha-\mu}(x+h-t)^{-\alpha}dt=$$
$$=C(x+h)^{\lambda}\int_0^{\frac{h}{x+h}}\xi^{-\alpha}(1-\xi)^{\lambda+\alpha-\mu}d\xi\le C_1h^{\lambda}\,,$$

where

$$C_1=C\sup_{0<\varepsilon<1}\varepsilon^{-\lambda}\int_0^{\varepsilon}\xi^{-\alpha}(1-\xi)^{\lambda+\alpha-\mu}d\xi<\infty.$$

The estimate of F_2 in the case $1\le\mu<\lambda+1$. Applying the estimate (11) for $D(x,h,t)$, we obtain

$$|F_2|\le Ch(x+h)^{\mu-1}\int_0^{x}\frac{t^{\lambda+\alpha-\mu}dt}{(x+h-t)(x-t)^{\alpha}}=$$
$$=Ch(x+h)^{\lambda-1}\int_0^{\frac{x}{x+h}}\frac{\xi^{\lambda+\alpha-\mu}d\xi}{(1-\xi)\left(\frac{x}{x+h}-\xi\right)^{\alpha}}\le C_2h^{\lambda}\,,$$

where

$$C_2=C\sup_{0<\varepsilon<1}\varepsilon^{1-\lambda}\int_0^{1-\varepsilon}\frac{\xi^{\lambda+\alpha-\mu}d\xi}{(1-\xi)(1-\varepsilon-\xi)^{\alpha}}<\infty.$$

3. Main result

The Riemann – Liouville mixed fractional integration operator has a form

$$\left(I_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{x}\int_0^{y}\frac{\varphi(t,s)dtds}{(x-t)^{1-\alpha}(y-s)^{1-\beta}}\,, \qquad (15)$$

where $x,y>0$, $0<\alpha,\beta<1$.

The corresponding mixed fractional differentiation operators are introduced in the Marchaud form

$$\left(\mathrm{D}_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left(\frac{\varphi(x,y)}{x^{\alpha}y^{\beta}}+\right.$$
$$\left.+\alpha\beta\int_0^{x}\int_0^{y}\frac{\varphi(x,y)-\varphi(t,s)}{(x-t)^{1+\alpha}(y-s)^{1+\beta}}dtds\right),\ x>0,y>0 \qquad (16)$$

in the case $0<\alpha,\beta<1$.

The definition in the Marchaud form may be used for all $-1<\alpha,\beta<1$: if $\alpha,\beta>0$ (16) gives the mixed fractional derivative, if $\alpha,\beta<0$, it is mixed fractional integral.

We shall use the united notation

$$J_{0+,0+}^{\alpha,\beta}=\begin{cases}I_{0+,0+}^{\alpha,\beta},\ \text{if}\ \ 0\le\alpha,\beta<1\\ \mathrm{D}_{0+,0+}^{-\alpha,-\beta},\ \text{if}\ \ 0\le\alpha,\beta<1\end{cases} \qquad (17)$$

Let $\rho(x,y)=\rho(x)\rho(y)$ be the weight function and put $\psi(x,y)=\rho(x,y)\varphi(x,y)$, $\varphi(x,y)\in H_0^{\lambda,\gamma}(Q;\rho)$. Evidently $\psi\in H^{\lambda,\gamma}(Q)$ and $\psi(x,y)|_{x=0,y=0}=0$. It is easy to see that

$$\left(\rho J_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\left(J_{0+,0+}^{\alpha,\beta}\psi\right)(x,y)+C\left(K_{0+,0+}^{\alpha,\beta}\psi\right)(x,y), \qquad (18)$$

$-1<\alpha,\ \beta<1$, where C=*const* and

$$\left(K_{0+,0+}^{\alpha,\beta}\psi\right)(x,y)=\int_0^{x}\int_0^{y}B(x,y;t,s)\psi(t,s)dtds\,, \qquad (19)$$

so that in (18) we have the mixed frational integral if $0\le\alpha,\beta<1$ and the mixed fractional derivative if $-1<\alpha,\beta\le0$.

The representation (13) for the fractional (integral) derivative shows that the estimate for the continuity modulus in the weighted case reduces to two simpler estimates:

1) the known non-weighted estimate of Hardy-Littlewood for mixed fractional integral (see [8], [9], [12], [15], [16]) and mixed fractional derivative (see [9], [10], [11], [12]); in the case weighted estimated of Hardy-Littlewood for mixed fractional integral (see [8]);

2) the estimate of the second term in (18), which is the mixed fractional derivative. It is the main part of the job.

Let

$$\rho(x,y)=x^{\mu}y^{\nu}\,,\qquad \mu<1+\lambda,\quad \nu<\gamma+1. \qquad (16)$$

Theorem 2. Let $\alpha,\beta\in(0,1)$, $\lambda,\gamma\in(0,1)$, $\rho(x,y)=x^{\mu}y^{\nu}$ and $|\psi(x,y)|\le Cx^{\lambda+\alpha}y^{\gamma+\beta}$ with $\mu<1+\lambda$, $\nu<\gamma+1$. Then the operator $K_{0+,0+}^{-\alpha,-\beta}$ is bounded from the space $\widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(\rho)$ into $\widetilde{H}_0^{\lambda,\gamma}(\rho)$.

Proof. To estimate the term $\left(K_{0+,0+}^{\alpha,\beta}\psi\right)(x,y)$, we note that the weight being degenerate, we have

$$\rho(x,y)-\rho(t,s)=[\rho(x)-\rho(t)][\rho(y)-\rho(s)]+\rho(s)[\rho(x)-\rho(t)]+$$
$$+\rho(t)[\rho(y)-\rho(s)].$$

This leads to the following representation

$$\left(K_{0+,0+}^{\alpha,\beta}\psi\right)(x,y)=G_2(x,y)=\int_0^x\int_0^y B_1(x,t)B_2(y,s)\psi(t,s)dtds+$$
$$+\int_0^x\int_0^y B_1(x,t)\frac{\psi(t,s)dtds}{(y-s)^{1+\beta}}+\int_0^x\int_0^y B_2(y,s)\frac{\psi(t,s)dtds}{(x-t)^{1+\alpha}},$$

where the notation (9) has been used. For the difference $G_2(x+h,y)-G_2(x,y)=\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)$ with $h>0$ and $x,x+h\in(0,b)$, we have

$$\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)=\int_x^{x+h}\int_0^y B_1(x+h,t)B_2(y,s)\psi(t,s)dtds+$$
$$+\int_0^x\int_0^y D_1(x,h,t)B_2(y,s)\psi(t,s)dtds+$$
$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\frac{\psi(t,s)}{(y-s)^{1+\beta}}dtds+$$
$$+\int_0^x\int_0^y D_1(x+h,t)\frac{\psi(t,s)}{(y-s)^{1+\beta}}dtds+$$
$$+\int_x^{x+h}\int_0^y B_2(y,s)\frac{\psi(t,s)}{(x+h-t)^{1+\alpha}}dtds+$$
$$+\int_0^x\int_0^y B_2(y,s)\left[\frac{1}{(x+h-t)^{1+\alpha}}-\frac{1}{(x-t)^{1+\alpha}}\right]\psi(t,s)dtds.$$

Since $\psi\in\widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}$, we have

$$|\psi(t,s)|\le Ct^{\lambda+\alpha}s^{\beta+\gamma},\quad |\psi(t,s)-\psi(x,0)|\le C(t-x)^{\lambda+\alpha}s^{\gamma+\beta}.$$

Then

$$\left|\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)\right|\le C\left\{\left(\int_x^{x+h}|B_1(x+h,t)|t^{\lambda+\alpha}dt+\int_0^x|D_1(x,h,t)|t^{\lambda+\alpha}dt+\right.\right.$$
$$\left.+\int_x^{x+h}\frac{(t-x)^{\lambda+\alpha}dt}{(x+h-t)^{1+\alpha}}+\int_0^x\left|\frac{1}{(x+h-t)^{1+\alpha}}-\frac{1}{(x-t)^{1+\alpha}}\right|(t-x)^{\lambda+\alpha}dt\right)\times$$
$$\times y^{\nu-1}\int_0^y\frac{(y-s)^{\gamma}}{s^{\nu}}ds+\left(\int_x^{x+h}|B_1(x+h,t)|t^{\lambda+\alpha}dt+\right.$$
$$\left.\left.+\int_0^x|D_1(x,h,t)|t^{\lambda+\alpha}\right)\int_0^y\frac{s^{\gamma+\beta}ds}{(y-s)^{1+\beta}}\right\}.$$

Hence, by estimates for F_1 and F_2 from Theorem 1, we have

$$\left|\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)\right|\le C_1h^{\lambda}.$$

The estimate

$$\left|\left(\overset{0,1}{\Delta}_\eta G_2\right)(x,y)\right|\le C_2\eta^{\gamma}$$

is symmetrical obtained.

For the mixed difference $\left(\overset{1,1}{\Delta}_{h,\eta} G_2\right)(x,y)$ with $h,\eta>0$ and $x,x+h\in[0,b]$, $y,y+\eta\in[0,d]$ the appropriate representation leading to the separate evaluation in each variable without losses in another variable is as follows:

$$\left(\overset{1,1}{\Delta}_{h,\eta} G_2\right)(x,y)=\int_x^{x+h}\int_y^{y+\eta}B_1(x+h,t)B_2(y+\eta,s)\psi(t,s)dtds+$$
$$+\int_0^x\int_0^y D_1(x,h,t)D_2(y,\eta,s)\psi(t,s)dtds+$$
$$+\int_x^{x+h}\int_0^y B_1(x+h,t)D_2(y,\eta,s)\psi(t,s)dtds+$$
$$+\int_0^x\int_y^{y+\eta}D_1(x,h,t)B_2(y+\eta,s)\psi(t,s)dtds+$$
$$+\int_x^{x+h}\int_y^{y+\eta}\frac{B_1(x+h,t)}{(y+\eta-s)^{1+\beta}}\psi(t,s)dtds+$$
$$+\int_x^{x+h}\int_y^{y+\eta}\frac{B_2(y+\eta,s)}{(x+h-t)^{1+\alpha}}\psi(t,s)dtds+$$
$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\left[\frac{1}{(y+\eta-s)^{1+\beta}}-\frac{1}{(y-s)^{\beta+1}}\right]\psi(t,s)dtds+$$
$$+\int_0^x\int_0^y D_1(x,h,t)\left[\frac{1}{(y+\eta-s)^{1+\beta}}-\frac{1}{(y-s)^{\beta+1}}\right]\psi(t,s)dtds+$$
$$+\int_0^x\int_0^y D_2(y,\eta,t)\left[\frac{1}{(x+h-t)^{1+\alpha}}-\frac{1}{(x-t)^{\alpha+1}}\right]\psi(t,s)dtds+$$
$$+\int_0^x\int_y^{y+\eta}D_1(x,h,t)\frac{\psi(t,s)}{(y+\eta-s)^{1+\beta}}dtds+$$
$$+\int_x^{x+h}\int_0^y D_2(y,\eta,s)\frac{\psi(t,s)}{(x+h-t)^{1+\alpha}}dtds+$$
$$+\int_0^x\int_y^{y+\eta}B_2(y+\eta,s)\left[\frac{1}{(x+h-t)^{1+\alpha}}-\frac{1}{(x-t)^{\alpha+1}}\right]\psi(t,s)dtds.$$

We omit the details of evaluation of each term in the above representation; it is standard via Lemma 1 and yields

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} G_2\right)(x,y)\right|\le C_3h^{\lambda}\eta^{\gamma}.$$

This completes the proof.

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